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# The isoscalar factors of $O_6 \times O_6$

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**Abstract.** Using the tensor basis and utilising the Wigner-Eckart theorem, we obtain the matrix representations $\langle\mu 00\rangle$ and $\langle\mu 10\rangle$ of $O_6(S\,T) \supset O_3(S) \times O_3(T)$. The isoscalar factors of $O_6(S\,T) \times O_6(S\,T) \supset O_6(S\,T) \supset O_3(S) \times O_3(T)$ are also calculated.

## 1. Introduction

Recently the extended interacting boson model of light nuclei IBM4 has been discussed (Elliott and White 1980, Elliott and Evans 1981, Halse *et al* 1984). A possible example with O(6) dynamical symmetry has been given, which includes the even-even nucleus $^{30}$Si and odd-odd nucleus $^{30}$P in a multiplet (Han *et al* 1987). However to discuss the IBM4 model further, for example to discuss the $\gamma$-transitions or the particle-transfer reactions, then the *wavefunctions* or the reduction coefficients of the dynamical symmetry group chain are needed.

The three medium coupling group chains of IBM4 all include the subgroup chain $O_6(ST) \supset O_3(S) \times O_3(T)$:

$$U_{36} \supset U_6(s\,d) \times U_6(S\,T) \supset U_5(d) \times O_6(S\,T) \supset O_5(d) \times O_3(S) \times O_3(T) \supset \ldots$$

$$U_{36} \supset U_6(s\,d) \times U_6(S\,T) \supset O_6(s\,d) \times O_6(S\,T) \supset O_5(d) \times O_3(S) \times O_3(T) \supset \ldots$$

$$U_{36} \supset U_6(s\,d) \times U_6(S\,T) \supset SU_3(s\,d) \times O_6(S\,T) \supset O_3(d) \times O_3(S) \times O_3(T) \supset \ldots.$$

This $O_6(S\,T)$ of bosons is isomorphic to the Wigner supermultiplet group SU(4) of nucleons at the Lie algebraic level. In the lowest approximation of IBM4, only the totally symmetric representations of $U_6(s\,d)$ are important. The reduction coefficients related to $U_6(s\,d)$ and its subgroups have been given in the program PHINT (by Scholten). So what we need is only the isoscalar factors (ISF) of $O_6(S\,T) \times O_6(S\,T) \supset O_6(S\,T) \supset O_3(S) \times O_3(T)$. This is the direct motivation to calculate the ISF.

We start from the boson realisation of $O_6(ST)$ and write its generators as irreducible tensor operators of $O_3(S)$ and $O_3(T)$. We obtain the totally symmetric irreducible representations (IR) $\langle\mu\ 0\ 0\rangle$ of $O_6(S\,T)$ by utilising the Wigner-Eckart theorem. Then we derive the ISF of $\langle\mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle = \langle\mu+1\ 0\ 0\rangle \oplus \langle\mu-1\ 0\ 0\rangle \oplus \langle\mu\ 1\ 0\rangle$ and at the same time we obtain the IR $\langle\mu\ 1\ 0\rangle$. The IR $\langle\mu\ 1\ 0\rangle$ is not totally symmetric and is not simply reduced according to the group chain $O_6(S\,T) \supset O_3(S) \times O_3(T)$. The simply reduced ISF above have been given by Hecht and Pang (1969). Some Wigner supermultiplet bases have been discussed, including a canonical orthonormal one (Hecht *et al* 1987). Here we also propose a method for labelling the degenerate states. This is an interesting new application of the tensor basis method to obtain the IR of semisimple Lie algebras (Biedenharn 1963, Baird and Biedenharn 1963, 1964, Sun and Han 1965, 1981, Yang *et al* 1964).

The tensor basis of $O_6(S\,T)$ is given in § 2. The IR are given in § 3. The ISF from $\langle\mu\ 0\ 0\rangle\times\langle 1\ 0\ 0\rangle$ to $\langle\mu+1\ 0\ 0\rangle$ and $\langle\mu-1\ 0\ 0\rangle$ are given in § 4. The ISF from $\langle\mu\ 0\ 0\rangle\times\langle 1\ 0\ 0\rangle$ to $\langle\mu\ 1\ 0\rangle$ and the IR $\langle\mu\ 1\ 0\rangle$ are given in § 5.

## 2. Tensor basis of $O_6$

It is known that the Cartan–Weyl basis of $O_6$ is

$$H_i \qquad E_{\pm e_j \pm e_k} \qquad i, j, k = 1, 2, 3 \qquad j \neq k \tag{2.1}$$

where $e_1 = (1\,0\,0)$, $e_2 = (0\,1\,0)$, $e_3 = (0\,0\,1)$ form an orthonormal basis in $R^3$ and $\pm e_j \pm e_k$ are the roots of $O_6$.

Let $\xi_q^\dagger$, $\eta_q^\dagger$ and $\xi_q$, $\eta_q$ be creation and annihilation operators of two kinds of bosons with angular momentum one. $q$ is the quantum number of the $z$ component of the angular momentum, $q = 0, \pm 1$. Consider the following operators:

$$\begin{aligned} S_q &= \sqrt{2}(\xi^\dagger \tilde{\xi})^1_q = \sum_{q'q''} \xi^\dagger_{q'} \tilde{\xi}_{q''} \langle 1\ q'\ 1\ q'' | 1\ q\rangle \\ T_q &= \sqrt{2}(\eta^\dagger \tilde{\eta})^1_q = \sum_{q'q''} \eta^\dagger_{q'} \tilde{\eta}_{q''} \langle 1\ q'\ 1\ q'' | 1\ q\rangle \\ V_{q_1 q_2} &= i(\xi^\dagger_{q_1} \tilde{\eta}_{q_2} - \eta^\dagger_{q_2} \tilde{\xi}_{q_1}) \qquad q, q_1, q_2 = 0, \pm 1 \end{aligned} \tag{2.2}$$

where $\langle 1\ q'\ 1\ q'' | 1\ q\rangle$ are Clebsch–Gordan coefficients of $O_3$, and

$$\tilde{\xi}_q = (-1)^{1+q} \xi_{-q} \qquad \tilde{\eta}_q = (-1)^{1+q} \eta_{-q}. \tag{2.3}$$

When

$$\begin{aligned} &H_1 = S_0/2\sqrt{2} && H_2 = T_0/2\sqrt{2} \qquad H_3 = V_{00}/2\sqrt{2} \\ &E_{e_1 \pm e_2} = V_{1\pm1}/2\sqrt{2} && E_{-(e_1 \pm e_2)} = V_{-1\mp1}/2\sqrt{2} \\ &E_{e_1 \pm e_3} = (S_1 \pm V_{10})/4 && E_{-(e_1 \pm e_3)} = -(S_{-1} \pm V_{-10})/4 \\ &E_{e_2 \pm e_3} = (T_1 \pm V_{01})/4 && E_{-(e_2 \pm e_3)} = -(T_{-1} \pm V_{0-1})/4 \end{aligned} \tag{2.4}$$

by straightforward calculation we see the operators $H_i$, $E_{\pm e_j \pm e_k}$ generate the $O_6$ group. So we can use the operators $S_q$, $T_q$ and $V_{q_1 q_2}$ as the tensor basis of $O_6$. This is a boson realisation of $O_6(S\,T)$.

The commutation relations can be written as:

$$\begin{aligned} &[S_0, S_{\pm1}] = \pm S_{\pm1} && [S_{+1}, S_{-1}] = -S_0 \\ &[T_0, T_{\pm1}] = \pm T_{\pm1} && [T_{+1}, T_{-1}] = -T_0 \end{aligned} \tag{2.5}$$

$$\begin{aligned} &[S_0, V_{q_1 q_2}] = q_1 V_{q_1 q_2} \\ &[S_{\pm1}, V_{q_1 q_2}] = \mp[(1 \mp q_1)(1 \pm q_1 + 1)/2]^{1/2} V_{q_1 \pm 1\, q_2} \\ &[T_0, V_{q_1 q_2}] = q_2 V_{q_1 q_2} \\ &[T_1, V_{q_1 q_2}] = \mp[(1 \mp q_2)(1 \pm q_2 + 1)/2]^{1/2} V_{q_1\, q_2 \pm 1} \end{aligned} \tag{2.6}$$

and

$$(VV)^{10}_{q0} = \sqrt{3/2}\, S_q \qquad (VV)^{01}_{0q} = \sqrt{3/2}\, T_q$$
$$(VV)^{2\,1}_{q_1 q_2} = 0 \qquad (VV)^{1\,2}_{q_1 q_2} = 0 \tag{2.7}$$

where

$$(VV)^{k_1 k_2}_{q_1 q_2} = \sum_{\substack{q_1', q_2', q_1'', q_2'' \\ V_{q_1' q_2'}, V_{q_1'' q_2''}}} \langle 1\, q_1'\, 1\, q_1'' | k_1 q_1 \rangle \langle 1\, q_2'\, 1\, q_2'' | k_2\, q_2 \rangle. \tag{2.8}$$

In this boson realisation of $O_6(ST)$, the above commutation relations can be derived from the commutation relations of bosons

$$[\xi_q, \xi_{q'}] = [\eta_q, \eta_{q'}] = 0 \qquad [\xi_q, \eta_{q'}] = [\xi_q, \eta^\dagger_{q'}] = 0$$
$$[\xi_q, \xi^\dagger_{q'}] = [\eta_q, \eta^\dagger_{q'}] = \delta_{qq'}. \tag{2.9}$$

From (2.5) we see that two independent rotation groups $O_3(S)$ and $O_3(T)$ are generated by $S_q$ and $T_q$ respectively. From (2.6) we notice that $V_{q_1 q_2}$ are double irreducible tensor operators of $O_3(S)$ and $O_3(T)$, where $q_1$ and $q_2$ are tensor component indices of $O_3(S)$ and $O_3(T)$.

The Casimir operator of $O_6(S\,T)$ is

$$C_2 = S^2 + T^2 + 3(VV)^{00}_{00} \tag{2.10}$$

where

$$S^2 = -S_{+1}S_{-1} - S_{-1}S_{+1} + S_0^2 \qquad T^2 = -T_{+1}T_{-1} - T_{-1}T_{+1} + T_0^2.$$

## 3. Totally symmetric irreducible representations

In this section we show how the tensor bases $S_q$, $T_q$ and $V_{q_1 q_2}$ are convenient for deriving the IR of $O_6 \supset O_3(S) \times O_3(T)$. At first we give the known wavefunctions $|\chi_S\, S\, M_S\rangle_\xi$ and $|\chi_T\, T\, M_T\rangle_\eta$, which are classified by the group chains $U_3(S) \supset O_3(S) \supset O_2(S)$ and $U_3(T) \supset O_3(T) \supset O_2(T)$. $U_3(S) \times U_3(T)$ and $O_6$ are all subgroups of $U_6$. So the states in the space of totally symmetric representation $\langle \mu\ 0\ 0 \rangle$ can be written as the linear combination of the direct products $|\chi_S\, S\, M_S\rangle_\xi\, |\chi_T\, T\, M_T\rangle_\eta$.

The groups $U_3(S)$ and $U_3(T)$ are generated by $\xi^\dagger_q \xi_{q'}$ and $\eta^\dagger_q \eta_{q'}$ respectively, with $q, q' = 0, \pm 1$. Let $P^\dagger_\xi$ and $P^\dagger_\eta$ be the $\xi$-pair and $\eta$-pair creation operators

$$P^\dagger_\xi = \sqrt{3/2}(\xi^\dagger \xi^\dagger)^0_0 \qquad P^\dagger_\eta = \sqrt{3/2}\,(\eta^\dagger \eta^\dagger)^0_0. \tag{3.1}$$

$P^\dagger_\xi$ and $P^\dagger_\eta$ are $O_3(S)$ and $O_3(T)$ invariants:

$$[S_q, P^\dagger_\xi] = 0 \qquad [T_q, P^\dagger_\eta] = 0. \tag{3.2}$$

It is known that

$$|\chi_S\, S\, S\rangle_\xi = C(\chi_S\, S) P^{\dagger\, \chi_S/2}_\xi \xi_1^{\dagger S} |0\,0\,0\rangle_\xi$$
$$|\chi_T\, T\, T\rangle_\eta = C(\chi_T\, T) P^{\dagger\, \chi_T/2}_\eta \eta_1^{\dagger T} |0\,0\,0\rangle_\eta \tag{3.3}$$

where

$$C(x, y) = \left( \frac{(2y+1)!!}{(x/2)!\, y!\, (2y+x+1)!!} \right)^{1/2}. \tag{3.4}$$

In the wavefunction $|\chi_S\, S\, M_S\rangle_\xi$, $\chi_S/2$ is the $\xi$-pair number and $S$ and $M_S$ are the quantum numbers of $S^2$ and $S_0$. Similarly in $|\chi_T\, T M_T\rangle_\eta$, $\chi_T/2$ is the $\eta$-pair number and $T$ and $M_T$ are the quantum numbers of $T^2$ and $T_0$. The states $|\chi_S\, S\, M_S\rangle_\xi$, and $|\chi_T\, T M_T\rangle_\eta$ can be obtained by operating $S_{-1}$ and $T_{-1}$ on $|\chi_S\, S\, S\rangle_\xi$ and $|\chi_T\, T\, T\rangle_\eta$ successively.

On the tensor basis of $O_6$, the Cartan subalgebra basis can be chosen as $\{S_0, T_0, V_{00}\}$. The highest-weight state $|\mu_1\, \mu_2\, \mu_3\rangle_{\rm hw}$ of the IR $\langle\mu_1\, \mu_2\, \mu_3\rangle$ satisfies

$$\begin{aligned} &S_0|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} = \mu_1|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} && T_0|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} = \mu_2|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} \\ &V_{00}|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} = \mu_3|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} && V_{1\pm1}|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} = 0 \\ &(S_1 \pm V_{10})|\mu_1\mu_2\,\mu_3\rangle_{\rm hw} = 0 && (T_1 \pm V_{01})|\mu_1\,\mu_2\,\mu_3\rangle_{\rm hw} = 0. \end{aligned} \tag{3.5}$$

In general the states in the IR $\Gamma$ of $O_6(S\,T) \supset O_3(S) \times O_3(T)$ can be written as

$$\left|\begin{matrix} \Gamma \\ k(\chi)\, S M_S\, T M_T \end{matrix}\right\rangle.$$

Where $S, M_S$ and $T, M_T$ are the quantum numbers of $S^2$, $S_0$ and $T^2$, $T_0$. $k$ is a degenerate quantum number which does not appear when $\Gamma = \langle\mu\ 0\, 0\rangle$. $(\chi)$ is not an independent quantum number but sometimes it is useful. For the IR $\langle\mu\ 0\ 0\rangle$ the states are written as

$$\left|\begin{matrix} \langle\mu\ 0\, 0\rangle \\ (\chi)\ S\, M_S\, T M_T \end{matrix}\right\rangle \qquad \text{or} \qquad |(\chi)\ S\, M_S\, T M_T\rangle$$

and $\chi + S + T = \mu$. From (3.5) it is easy to find the highest-weight state

$$|\mu\ 0\ 0\rangle_{\rm hw} = |(0)\ \mu\ \mu\ 0\ 0\rangle = |0\ \mu\ \mu\rangle_\xi |0\ 0\ 0\rangle_\eta \tag{3.6}$$

When the operators $S_q$ and $T_q$ act on $|(\chi)\ S\, M_S\, T M_T\rangle$, only $M_S$ and $M_T$ are possibly changed. The matrix elements of $S_q$ and $T_q$ are known as the same in $O_3(S)$ and $O_3(T)$. When $V_{q_1q_2}$ operate on the state $|(\chi)\ S\, M_S\, T M_T\rangle$, the quantum numbers $S$, $T$ and $(\chi)$ are changed. So when $V_{q_1q_2}$ acts repeatedly on the highest-weight state $|\mu\ 0\ 0\rangle_{\rm hw}$, we obtain the states with different $S$ and different $T$ in $\langle\mu\ 0\ 0\rangle$. From the Wigner–Eckart theorem we only need to calculate the reduced matrix elements $\langle(\chi')\ S'\ T' \| V \| (\chi)\ S\ T\rangle$ of $V_{q_1q_2}$:

$$\begin{aligned} &\langle(\chi')\ S'\ M'_S\ T'\ M'_T | V_{q_1q_2} | (\chi)\ S\, M_S\, T M_T\rangle \\ &\quad = [(2S'+1)(2T'+1)]^{-1/2} \langle S\, M_S\ 1\ q_1 | S'\ M'_S\rangle \langle T M_T\ 1\ q_2 | T'\ M'_T\rangle \\ &\quad \times \langle(\chi')\ S'\ T' \| V \| (\chi)\ S\ T\rangle. \end{aligned} \tag{3.7}$$

Notice that

$$V^\dagger_{q_1q_2} = (-1)^{q_1+q_2} V_{-q_1-q_2} \tag{3.8}$$

and we have

$$\langle(\chi)\ S\ T \| V \| (\chi')\ S'\ T'\rangle = \langle(\chi')\ S'\ T' \| V \| (\chi)\ S\ T\rangle. \tag{3.9}$$

Define

$$\{V | (\chi)\ S\ T\rangle\}^{S'T'}_{M'_SM'_T} = \sum_{q_1q_2} \langle S\, M_S\ 1\ q_1 | S' M'_S\rangle \langle T M_T\ 1\ q_2 | T' M'_T\rangle V_{q_1q_2} | (\chi)\ S\, M_S\, T M_T\rangle.$$

We obtain the recursion formulae between states with different $S$ and $T$

$$\{V|(\chi)\,S\,T\rangle\}^{S-1\,T+1}_{S-1\,T+1}=[(2S-1)(2T+3)]^{-1/2} \times\langle(\chi)\,S-1\;T+1\|V\|(\chi)\,S\,T\rangle|(\chi)\,S-1\,S-1\;T+1\;T+1\rangle \quad (3.10)$$

$$\{V|(\chi)\,S\,T\rangle\}^{S-1\,T-1}_{S-1\,T-1}=[(2S-1)(2T-1)]^{-1/2} \times\langle(\chi+2)\,S-1\;T-1\|V\|(\chi)\,S\,T\rangle|(\chi+2)\,S-1\,S-1\;T-1\;T-1\rangle. \quad (3.11)$$

The coefficients

$$[(2S-1)(2T+3)]^{-1/2}\langle(\chi)\,S-1\;T+1\|V\|(\chi)\,S\,T\rangle$$

and

$$[(2S-1)(2T-1)]^{-1/2}\langle(\chi+2)\,S-1\;T-1\|V\|(\chi)\,S\,T\rangle$$

can be treated as the normalised constants of the states $|(\chi)\,S-1\;S-1\;T+1\;T+1\rangle$ and $|(\chi+2)\,S-1\;S-1\;T-1\;T-1\rangle$ respectively. Starting from the highest-weight state $|\mu\;0\;0\rangle_{\mathrm{hw}}$, using the recursion formulae we can calculate every $|(\chi)\,S\,S\,T\,T\rangle$ state in $\langle\mu\;0\;0\rangle$. By mathematical induction we get the states

$$|(\chi)\,S\,S\,T\,T\rangle=\sum_{\delta}A(\chi,S,T,\delta)|\chi-\delta\,S\,S\rangle_{\xi}|\delta\,T\,T\rangle_{\eta} \qquad \chi=0,2,4,\ldots \quad (3.12)$$

and the non-zero reduced matrix elements of $V_{q_1q_2}$

$$\langle(\chi)\,S-1\;T+1\|V\|(\chi)\,S\,T\rangle=\langle(\chi)\,S\,T\|V\|(\chi)\,S-1\;T+1\rangle =[S(2S+\chi+1)(T+1)(2T+\chi+3)]^{1/2} \quad (3.13)$$

$$\langle(\chi+2)\,S-1\;T-1\|V\|(\chi)\,S\,T\rangle=\langle(\chi)\,S\,T\|V\|(\chi+2)\,S-1\;T-1\rangle =[ST(\chi+2)(2\mu+2-\chi)]^{1/2} \quad (3.14)$$

where

$$A(\chi,S,T,\delta)=(-\mathrm{i})^{T}(-1)^{\delta/2} \times\left(\frac{(2\mu+2-\chi)!!\,\chi!!\,(2S+\chi+1)!!\,(2T+\chi+1)!!}{(2\mu+2)!!\,\delta!!\,(\chi-\delta)!!\,(2S+\chi+1-\delta)!!\,(2T+\delta+1)!!}\right)^{1/2}. \quad (3.15)$$

The reduction rule according to the group chain $O_6(S\,T)\supset O_3(S)\times O_3(T)$ can be obtained from (3.13) and (3.14):

$$\langle\mu\;0\;0\rangle=\sum_{\substack{S+T+\chi=\mu\\ \chi=0,2,4,\ldots}}\oplus\;(S\,T) \quad (3.16)$$

so $\langle\mu\;0\;0\rangle$ is simply reduced.

The eigenvalue of the Casimir operator in IR $\langle\mu\;0\;0\rangle$ is

$${}_{\mathrm{hw}}\langle\mu\;0\;0|C_{206}|\mu\;0\;0\rangle_{\mathrm{hw}}=\mu(\mu+4). \quad (3.17)$$

For example, the highest-weight state of IR $\langle 1\;0\;0\rangle$ is

$$|1\;0\;0\rangle_{\mathrm{hw}}=|(0)\;1\;1\;0\;0\rangle=|0\;1\;1\rangle_{\xi}|0\;0\;0\rangle_{\eta}.$$

The non-zero reduced matrix elements of $V$ are

$$\langle(0)\;0\;1\|V\|(0)\;1\;0\rangle=\langle(0)\;1\;0\|V\|(0)\;0\;1\rangle=3.$$

The states with $S=M_S=0$ and $T=M_T=1$ is

$$|(0)\,0\,0\,1\,1\rangle=-\mathrm{i}|0\,0\,0\rangle_\xi|0\,1\,1\rangle_\eta.$$

In IR $\langle 1\,0\,0\rangle$ there are only two states with $(S\,T)=(1\,0)$ and $(S\,T)=(0\,1)$.

## 4. The isoscalar factors (ISF) of $O_6(1)\times O_6(2)\supset O_6$

Consider two independent systems with $O_6(1)$ and $O_6(2)$ symmetry respectively. The generators are $S_q(i)$, $T_q(i)$ and $V_{q_1q_2}(i)$ for $O_6(i)$, $i=1,2$. These two systems form a coupled system with $O_6$ symmetry. The generators of $O_6$ are

$$S_q=S_q(1)+S_q(2)\qquad T_q=T_q(1)+T_q(2)\qquad V_{q_1q_2}=V_{q_1q_2}(1)+V_{q_1q_2}(2) \tag{4.1}$$

Suppose the states

$$\left|\begin{matrix}\Gamma_i\\ k_i\ (\chi_i)\ S_i\,M_{si}\ T_i\,M_{Ti}\end{matrix}\right\rangle$$

form a basis of the $\Gamma_i$ representation space of $O_6(i)$ and

$$\left|\begin{matrix}\Gamma\\ k\ (\chi)\ S\ M_S\ T\,M_T\end{matrix}\right\rangle$$

form a basis of the $\Gamma$ representation space of $O_6$. Then the ISF

$$\left\langle\begin{matrix}\Gamma_1 & & & \Gamma_2 & & \\ k_1 & S_1 & T_1 & k_2 & S_2 & T_2\end{matrix}\right|\left.\begin{matrix} \\ k(\chi)\ S\ T\end{matrix}\right\rangle$$

of $O_6(1)\times O_6(2)\supset O_6(S\,T)\supset O_3(S)\times O_3(T)$ are defined as

$$\begin{aligned}\left|\begin{matrix}\Gamma(\Gamma_1\Gamma_2)\\ k\ (\chi)S\,M_S\ T\ M_T\end{matrix}\right\rangle &= \sum_{k_1,S_1,T_1,k_2,S_2,T_2}\left\langle\begin{matrix}\Gamma_1 & & & \Gamma_2 & & \\ k_1 & S_1 & T_1 & K_2 & S_2 & T_2\end{matrix}\right|\left.\begin{matrix}\Gamma\\ k\ (\chi)\ S\ \ T\end{matrix}\right\rangle\\ &\quad\times\left|\begin{matrix}\Gamma_1 & & & \Gamma_2 & & \\ k_1\ (\chi_1) & S_1 & T_1 & k_2\ (\chi_2) & S_2 & T_2\end{matrix}\ S\,M_S\ T\,M_T\right\rangle\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}\left|\begin{matrix}\Gamma_1 & & & \Gamma_2 & & \\ k_1\ (\chi_1) & S_1 & T_1 & k_2\ (\chi_2) & S_2 & T_2\end{matrix}\ S\,M_S\ T\,M_T\right\rangle &= \sum_{M_{S1},M_{S2},M_{T1},M_{T2}}\langle S_1\,M_{S1}\ S_2\,M_{S2}|S\,M_S\rangle\langle T_1\,M_{T1}\ T_2\,M_{T2}|T\,M_T\rangle\\ &\quad\times\left|\begin{matrix}\Gamma_1\\ k_1\ (\chi_1)\ S_1\ M_{S1}\ T_1\ M_{T1}\end{matrix}\right\rangle\left|\begin{matrix}\Gamma_2\\ k_2\ (\chi_2)\ S_2\ M_{S2}\ T_2\ M_{T2}\end{matrix}\right\rangle.\end{aligned} \tag{4.3}$$

In this paper we only discuss the ISF with $\Gamma_1=\langle\mu\ 0\,0\rangle$ and $\Gamma_2=\langle 1\,0\,0\rangle$. From § 3 we know that in $\langle\mu\ 0\,0\rangle$ and $\langle 1\,0\,0\rangle$ representation spaces the states

$$\left|\begin{matrix}\Gamma_i\\ (\chi_i)\ S_i\ M_{Si}\ T_i\,M_{Ti}\end{matrix}\right\rangle$$

form an orthonormal basis and the degenerate quantum numbers $k_i$ are not needed. Then formulae (4.2) and (4.3) are simplified to

$$\left| \begin{matrix} \Gamma(\Gamma_1\Gamma_2) \\ k\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right\rangle = \sum_{S_1,T_1,S_2,T_2} \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S_1\ T_1 & S_2\ T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ k\ (\chi)\ S\ T \end{matrix} \right\rangle \times \left| \begin{matrix} \Gamma_1 & \Gamma_2 \\ (\chi_1)\ S_1\ T_1 & (\chi_2)\ S_2\ T_2 \end{matrix}\ S\ M_S\ T\ M_T \right\rangle \tag{4.4}$$

$$\left| \begin{matrix} \Gamma_1 & \Gamma_2 \\ (\chi_1)\ S_1\ T_1 & (\chi_2)\ S_2\ T_2 \end{matrix}\ S\ M_S\ T\ M_T \right\rangle = \sum_{M_{S1},M_{S2},M_{T1},M_{T2}} \langle S_1\ M_{S1}\ S_2\ M_{S2} | S\ M_S \rangle \langle T_1\ M_{T1}\ T_2\ M_{T2} | T\ M_T \rangle \times \left| \begin{matrix} \Gamma_1 \\ (\chi_1)\ S_1\ M_{S1}\ T_1\ M_{T1} \end{matrix} \right\rangle \left| \begin{matrix} \Gamma_2 \\ (\chi_2)\ S_2\ M_{S2}\ T_2\ M_{T2} \end{matrix} \right\rangle. \tag{4.5}$$

The states

$$\left| \begin{matrix} \Gamma_1 & \Gamma_2 \\ (\chi_1)\ S_1\ T_1 & (\chi_2)\ S_2\ T_2 \end{matrix}\ S\ M_S\ T\ M_T \right\rangle$$

also from an orthonormal basis for $O_6(1) \times O_6(2) \supset O_6$. Then we get

$$\left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S_1\ T_1 & S_2\ T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ k\ (\chi)\ S\ T \end{matrix} \right\rangle = \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ (\chi_1)\ S_1\ T_1 & (\chi_2)\ S_2\ T_2 \end{matrix}\ S\ M_S\ T\ M_T \middle| \begin{matrix} \Gamma \\ k\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right\rangle. \tag{4.6}$$

We notice that the right-hand side of (4.6) is independent of $M_S$ and $M_T$.

In general we need not choose an orthonormal basis for representation space. For example, it is convenient to choose the $\langle \mu\ 1\ 0 \rangle$ basis

$$\left| \begin{matrix} \langle \mu\ 1\ 0 \rangle \\ k\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right\rangle$$

which are normalised but are not orthogonal for different $k$. Of course they are still orthogonal for different $S$, $M_S$, $T$ and $M_T$. So we introduce a metric tensor $(\rho_{ij})$ in the subspace of $\Gamma$ with definite $S$ and $T$:

$$\begin{aligned} \rho_{ij}(S\ T) &= \left\langle \begin{matrix} \Gamma \\ i\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \middle| \begin{matrix} \Gamma \\ j\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right\rangle \\ \rho_{ii}(S\ T) &= 1. \end{aligned} \tag{4.7}$$

Suppose $(\rho^{ij})$ is the inverse of $(\rho_{ij})$. Then the identity operator in $\Gamma$ representation space can be written as

$$\sum_{i,j,S,M_S,T,M_T} \rho^{ij} \left| \begin{matrix} \Gamma \\ i\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right\rangle \left\langle \begin{matrix} \Gamma \\ j\ (\chi)\ S\ M_S\ T\ M_T \end{matrix} \right| = I. \tag{4.8}$$

Using (4.4), (4.5) and (4.8) we obtain the recursion formula of the ISF

$$\sum_{i,j} \rho^{ij}(S\,T)\left\langle \begin{matrix} \Gamma \\ j\,(\chi')\,S'\,T' \end{matrix} \middle\| V \middle\| \begin{matrix} \Gamma \\ k\,(\chi)\,S\,T \end{matrix} \right\rangle \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S'_1\,T'_1 & S'_2\,T'_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ i\,(\chi')\,S'\,T' \end{matrix} \right\rangle$$

$$= \sum_{S_1,T_1,S_2,T_2} \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi'_1)\,S'_1\,T'_1 & (\chi'_2)\,S'_2\,T'_2 & S'\,T' \end{matrix} \middle\| [V(1)+V(2)] \middle\| \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi_1)\,S_1\,T_1 & (\chi_2)\,S_2\,T_2 & S\,T \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S_1\,T_1 & S_2\,T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ k\,(\chi)\,S\,T \end{matrix} \right\rangle \tag{4.9}$$

where

$$\left\langle \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi'_1)\,S'_1\,T'_1 & (\chi'_2)\,S'_2\,T'_2 & S'\,T' \end{matrix} \middle\| [V(1)+V(2)] \middle\| \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi_1)\,S_1\,T_1 & (\chi_2)\,S_2\,T_2 & S\,T \end{matrix} \right\rangle$$

$$= f(S'_1, S'_2, S', S_1, S_2, S) f(T'_1, T'_2, T', T_1, T_2, T)$$

$$\times \left\langle \begin{matrix} \Gamma_1 \\ (\chi'_1)\,S'_1\,T'_1 \end{matrix} \middle\| V(1) \middle\| \begin{matrix} \Gamma_1 \\ (\chi_1)\,S_1\,T_1 \end{matrix} \right\rangle$$

$$+ g(S'_1, S'_2, S', S_1, S_2, S) g(T'_1, T'_2, T', T_1, T_2, T)$$

$$\times \left\langle \begin{matrix} \Gamma_2 \\ (\chi'_2)\,S'_2\,T'_2 \end{matrix} \middle\| V(2) \middle\| \begin{matrix} \Gamma_2 \\ (\chi_2)\,S_2\,T_2 \end{matrix} \right\rangle$$

and

$$f(S'_1, S'_2, S', S_1, S_2, S)$$

$$= \delta_{S_2 S'_2}(-1)^{S'_1+S'_2+S+1}[(2S'+1)(2S+1)]^{1/2} \begin{Bmatrix} 1 & S'_1 & S_1 \\ S'_2 & S & S' \end{Bmatrix}$$

$$g(S'_1, S'_2, S', S_1, S_2, S) = \delta_{S_1 S'_1}(-1)^{S'_1+S_2+S'+1}[(2S'+1)(2S+1)]^{1/2} \begin{Bmatrix} 1 & S'_2 & S_2 \\ S'_1 & S & S' \end{Bmatrix}$$

where the { } terms are 6-$j$ symbols. The ISF satisfy the normalisation condition

$$\sum_{S_1,T_1,S_2,T_2} \left| \left\langle \begin{matrix} \Gamma \\ k\,(\chi)\,S\,T \end{matrix} \middle| \begin{matrix} \Gamma_1 & \Gamma_2 \\ S_1\,T_1 & S_2\,T_2 \end{matrix} \right\rangle \right|^2 = 1. \tag{4.10}$$

When $\Gamma$ is a totally symmetric representation, we have

$$(\rho_{ij}) = (\rho^{ij}) = I. \tag{4.11}$$

The recursion formula (4.9) is simplified to

$$\left\langle \begin{matrix} \Gamma \\ (\chi')\,S'\,T' \end{matrix} \middle\| V \middle\| \begin{matrix} \Gamma \\ (\chi)\,S\,T \end{matrix} \right\rangle \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S'_1\,T'_1 & S'_2\,T'_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ (\chi')\,S'\,T' \end{matrix} \right\rangle$$

$$= \sum_{S_1,T_1,S_2,T_2} \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi'_1)\,S'_1\,T'_1 & (\chi'_2)\,S'_2\,T'_2 & S'\,T' \end{matrix} \middle\| [V(1)+V(2)] \middle\| \begin{matrix} \Gamma_1 & \Gamma_2 & \\ (\chi_1)\,S_1\,T_1 & (\chi_2)\,S_2\,T_2 & S\,T \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} \Gamma_1 & \Gamma_2 \\ S_1\,T_1 & S_2\,T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ (\chi)\,S\,T \end{matrix} \right\rangle. \tag{4.9$'$}$$

Suppose the direct product $\langle \mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle$ can be decomposed to the direct sum of irreducible representations $\langle \mu_1\ \mu_2\ \mu_3\rangle$:

$$\langle \mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle = \sum_{\mu_1,\mu_2,\mu_3} \oplus\ \langle \mu_1\ \mu_2\ \mu_3\rangle. \tag{4.12}$$

**Table 1.** ISF $\left\langle \begin{matrix} \langle \mu\, 0\, 0\rangle & \langle 1\, 0\, 0\rangle \\ (\chi_1)\, S_1\, T_1 & S_2\, T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ k\,(\chi)\, S\, T \end{matrix} \right\rangle$ of $\langle \mu\, 0\, 0\rangle \times \langle 1\, 0\, 0\rangle$ when $\chi$ is even.

| $\left\| \begin{matrix} \Gamma \\ k\,(\chi)\, S\, T \end{matrix} \right\rangle$ | $(\chi_1)\, S_1\, T_1\, S_2\, T_2$ | | | |
|---|---|---|---|---|
| | $(\chi)\, S-1\, T\, 1\, 0$ | $(\chi-2)\, S+1\, T\, 1\, 0$ | $(\chi)\, S\, T-1\, 0\, 1$ | $(\chi-2)\, S\, T+1\, 0\, 1$ |
| $\left\| \begin{matrix} \langle \mu+1\, 0\, 0\rangle \\ (\chi)\, S\, T \end{matrix} \right\rangle$ | $\left(\dfrac{S(2S+\chi+1)(2\mu+4-\chi)}{2(\mu+1)(\mu+2)(2S+1)}\right)^{1/2}$ | $\left(\dfrac{(S+1)\chi(2T+\chi+1)}{2(\mu+1)(\mu+2)(2S+1)}\right)^{1/2}$ | $\left(\dfrac{T(2T+\chi+1)(2\mu+4-\chi)}{2(\mu+1)(\mu+2)(2T+1)}\right)^{1/2}$ | $\left(\dfrac{(T+1)\chi(2S+\chi+1)}{2(\mu+1)(\mu+2)(2T+1)}\right)^{1/2}$ |
| $\left\| \begin{matrix} \langle \mu-1\, 0\, 0\rangle \\ (\chi)\, S\, T \end{matrix} \right\rangle$ | $-\left(\dfrac{S\chi(2T+\chi+1)}{2(\mu+2)(\mu+3)(2S+1)}\right)^{1/2}$ | $-\left(\dfrac{(S+1)(2S+\chi+1)(2\mu+4-\chi)}{2(\mu+2)(\mu+3)(2S+1)}\right)^{1/2}$ | $\left(\dfrac{T\chi(2S+\chi+1)}{2(\mu+2)(\mu+3)(2T+1)}\right)^{1/2}$ | $\left(\dfrac{(T+1)(2T+\chi+1)(2\mu+4-\chi)}{2(\mu+2)(\mu+3)(2T+1)}\right)^{1/2}$ |
| $\left\| \begin{matrix} \langle \mu\, 1\, 0\rangle \\ 1\,(\chi)\, S\, T \end{matrix} \right\rangle$ | $\left(\dfrac{(S+1)(T+1)\chi(2\mu+4-\chi)}{W(\chi, S, T)(2S+1)}\right)^{1/2}$ | $-\left(\dfrac{S(T+1)(2S+\chi+1)(2T+\chi+1)}{W(\chi, S, T)(2S+1)}\right)^{1/2}$ | 0 | $-\left(\dfrac{S(S+1)(2T+1)}{W(\chi, S, T)}\right)^{1/2}$ |
| $\left\| \begin{matrix} \langle \mu\, 1\, 0\rangle \\ 2\,(\chi)\, S\, T \end{matrix} \right\rangle$ | 0 | $-\left(\dfrac{T(T+1)(2S+1)}{W(\chi, T, S)}\right)^{1/2}$ | $\left(\dfrac{(S+1)(T+1)\chi(2\mu+4-\chi)}{W(\chi, T, S)(2T+1)}\right)^{1/2}$ | $-\left(\dfrac{(S+1)T(2S+\chi+1)(2T+\chi+1)}{W(\chi, T, S)(2T+1)}\right)^{1/2}$ |

The highest-weight state of $\langle\mu_1\,\mu_2\,\mu_3\rangle$ is $|\mu_1\,\mu_2\,\mu_3\rangle_{\text{hw}}$. Using (3.5), (3.13), (3.14) and (4.1) we obtain

$$\mu_3=0 \qquad |\mu_1\,\mu_2\,0\rangle_{\text{hw}}=\left|\begin{matrix}\langle\mu_1\,\mu_2\,0\rangle\\ \mu_1\,\mu_1\,\mu_2\,\mu_2\end{matrix}\right\rangle \tag{4.13}$$

i.e. $S=M_S=\mu_1$ and $T=M_T=\mu_2$ for $|\mu_1\,\mu_2\,0\rangle_{\text{hw}}$. From (4.9) we obtain the ISF of $|\mu_1\,\mu_2\,0\rangle_{\text{hw}}$ satisfying

$$\sum_{S_1,T_1,S_2,T_2}\left\langle\begin{matrix}\langle\mu\,0\,0\rangle & \langle 1\,0\,0\rangle\\ (\chi_1')\,S_1'\,T_1' & (\chi_2')\,S_2'\,T_2'\end{matrix}\,\mu_1'\,\mu_2'\right\|[V(1)+V(2)]\left\|\begin{matrix}\langle\mu\,0\,0\rangle & \langle 1\,0\,0\rangle\\ (\chi_1)\,S_1\,T_1 & (\chi_2)\,S_2\,T_2\end{matrix}\,\mu_1\,\mu_2\right\rangle$$
$$\times\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ S_1\,T_1\quad S_2\,T_2\end{matrix}\right|\left.\begin{matrix}\langle\mu_1\,\mu_2\,0\rangle\\ (0)\,\mu_1\,\mu_2\end{matrix}\right\rangle=0 \tag{4.14}$$

where $\mu_1'=\mu_1+1$, $\mu_2'=\mu_2+1$ or $\mu_1'=\mu_1+1$, $\mu_2'=\mu_2$ or $\mu_1'=\mu_1$, $\mu_2'=\mu_2+1$.

It is proved there are three and only three independent solutions of (4.14):

$$\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ \mu\,0\quad 1\,0\end{matrix}\right|\left.\begin{matrix}\langle\mu+1\,0\,0\rangle\\ \mu+1\,0\end{matrix}\right\rangle=1$$
$$\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ \mu-2\,0\quad 1\,0\end{matrix}\right|\left.\begin{matrix}\mu-1\,0\,0\rangle\\ \mu-1\,0\end{matrix}\right\rangle=-\left[\frac{3(\mu-1)}{(\mu+2)(\mu+3)(2\mu-1)}\right]^{1/2} \tag{4.15}$$

$$\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ \mu\,0\quad 1\,0\end{matrix}\right|\left.\begin{matrix}\langle\mu-1\,0\,0\rangle\\ \mu-1\,0\end{matrix}\right\rangle=\left[\frac{\mu(\mu+1)(2\mu+1)}{(\mu+2)(\mu+3)(2\mu-1)}\right]^{1/2}$$
$$\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ \mu-1\,1\quad 0\,1\end{matrix}\right|\left.\begin{matrix}\langle\mu-1\,0\,0\rangle\\ \mu-1\,0\end{matrix}\right\rangle=\left[\frac{3(\mu+1)}{(\mu+2)(\mu+3}\right]^{1/2} \tag{4.16}$$

$$\left\langle\begin{matrix}\langle\mu\,0\,0\rangle\,\langle 1\,0\,0\rangle\\ \mu\,0\quad 0\,1\end{matrix}\right|\left.\begin{matrix}\langle\mu\,1\,0\rangle\\ \mu\,1\end{matrix}\right\rangle=1. \tag{4.17}$$

Thus we obtain the decomposition rule

$$\langle\mu\,0\,0\rangle\times\langle 1\,0\,0\rangle=\langle\mu+1\,0\,0\rangle\langle\mu-1\,0\,0\rangle\oplus\langle\mu\,1\,0\rangle. \tag{4.18}$$

Using the recursion formula (4.9) we obtain the ISF of $\langle\mu+1\,0\,0\rangle$ and $\langle\mu-1\,0\,0\rangle$ from (4.15) and (4.16). The results are given in table 1. Where $\chi+S+T=\mu+1$, $\chi=0,2,4,\ldots$ for $\langle\mu+1\,0\,0\rangle$ and $\chi=2,4,\ldots$ for $\langle\mu-1\,0\,0\rangle$.

## 5. Representations and ISF of $\langle\mu\,1\,0\rangle$

Notice that the states

$$\left|\begin{matrix}\langle\mu\,0\,0\rangle & \langle 1\,0\,0\rangle\\ (\chi_1)\,S_1\,T_1 & (\chi_2)\,S_2\,T_2\end{matrix}\,S\,M_S\,T\,M_T\right\rangle$$

form an orthonormal basis of $\langle\mu\,0\,0\rangle\times\langle 1\,0\,0\rangle$. From (4.18) and the results of § 3 we get the decomposition rule of $\langle\mu\,1\,0\rangle$ to $(S\,T)$ as

$$\begin{aligned}\langle\mu\,1\,0\rangle=(\mu\,1)&+(\mu-1\,2)+(\mu-2\,3)+\ldots+(1\,\mu)\\ &(\mu\,0)+(\mu-1\,1)^2+(\mu-2\,2)^2+\ldots+(1\,\mu-1)^2+(0\,\mu)\\ &\quad+(\mu-1\,0)+(\mu-2\,1)^2+\ldots+(1\,\mu-2)^2+(0\,\mu-1)\\ &\qquad(\mu-2\,0)+\ldots+(1\,\mu-3)^2+(0\,\mu-2)\\ &\qquad\quad+\ldots\\ &\qquad\qquad+(1\,0)\quad+(0\,1).\end{aligned} \tag{5.1}$$

The superscripts 2 of $(S\,T)$ indicate that $\langle \mu\ 1\ 0\rangle$ includes states with $(S\,T)$ twice. We use a degenerate quantum number for distinguishing the two states

$$\left| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ 1\ (\chi)\ S\ T \end{matrix} \right\rangle \quad \text{and} \quad \left| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ 2\ (\chi)\ S\ T \end{matrix} \right\rangle .$$

We also have $\chi + S + T = \mu + 1$. But in $\langle \mu\ 1\ 0\rangle$, $\chi$ can be even or odd.

When $\chi = 0$, $\langle \mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle$ is reduced to the singlet states

$$\left| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ (0)\ S\ T \end{matrix} \right\rangle$$

of $\langle \mu\ 1\ 0\rangle$. Considering the states

$$\left| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ (0)\ S\ T \end{matrix} \right\rangle$$

are orthogonal to the states

$$\left| \begin{matrix} \langle \mu+1\ 0\ 0\rangle \\ (0)\ S\ T \end{matrix} \right\rangle$$

from (4.9) we obtain the ISF

$$\begin{aligned} \left\langle \begin{matrix} \langle \mu\ 0\ 0\rangle & \langle 1\ 0\ 0\rangle \\ S-1\ T & 1\ 0 \end{matrix} \middle| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ (0)\ S\ T \end{matrix} \right\rangle &= \left[ \frac{T}{\mu+1} \right]^{1/2} \\ \left\langle \begin{matrix} \langle \mu\ 0\ 0\rangle & \langle 1\ 0\ 0\rangle \\ S\ T-1 & 0\ 1 \end{matrix} \middle| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ (0)\ S\ T \end{matrix} \right\rangle &= -\left[ \frac{S}{\mu+1} \right]^{1/2} . \end{aligned} \tag{5.2}$$

When $\chi$ is odd, $\langle \mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle$ can only be reduced to $\langle \mu\ 1\ 0\rangle$ . By suitable choice of the phase factors, we obtain the ISF in table 2. In this case the states

$$\left| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ k\ (\chi = \text{odd})\ S\ T \end{matrix} \right\rangle$$

are orthonormal and $\rho_{ij} = \rho^{ij} = \delta_{ij}$. In table 2 we see that when $T = 0$ or $S = 0$:

$$\begin{aligned} \left\langle \begin{matrix} \langle \mu\ 0\ 0\rangle & \langle 1\ 0\ 0\rangle \\ (\chi-1)\ S\ 0 & 1\ 0 \end{matrix} \middle| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ 1\ (\chi)\ S\ 0 \end{matrix} \right\rangle &= 1 \\ \left\langle \begin{matrix} \langle \mu\ 0\ 0\rangle & \langle 1\ 0\ 0\rangle \\ (\chi-1)\ 0\ T & 0\ 1 \end{matrix} \middle| \begin{matrix} \langle \mu\ 1\ 0\rangle \\ 2\ (\chi)\ 0\ T \end{matrix} \right\rangle &= 1. \end{aligned} \tag{5.3}$$

**Table 2.** ISF $\left\langle \begin{matrix} \langle \mu\ 0\ 0\rangle & \langle 1\ 0\ 0\rangle \\ (\chi_1)\ S_1\ T_1 & S_2\ T_2 \end{matrix} \middle| \begin{matrix} \Gamma \\ k\ (\chi)\ S\ T \end{matrix} \right\rangle$ of $\langle \mu\ 0\ 0\rangle \times \langle 1\ 0\ 0\rangle$ when $\chi$ is odd.

| | $(\chi_1)\ S_1\ T_1\ S_2\ T_2$ | |
|---|---|---|
| $\left\lvert \begin{matrix} \Gamma \\ k\ (\chi)\ S\ T \end{matrix} \right\rangle$ | $(x-1)\ S\ T\ 1\ 0$ | $(\chi-1)\ S\ T\ 0\ 1$ |
| $\left\lvert \begin{matrix} \mu\ 1\ 0 \\ 1\ (\chi)\ S\ T \end{matrix} \right\rangle$ | 1 | 0 |
| $\left\lvert \begin{matrix} \mu\ 1\ 0 \\ 2\ (\chi)\ S\ T \end{matrix} \right\rangle$ | 0 | 1 |

**Table 3.** Reduced matrix elements $\left\langle \begin{matrix} \langle\mu\, 1\, 0\rangle \\ i\,(\chi)\,S\,T \end{matrix} \middle\| V \middle\| \begin{matrix} \langle\mu\, 1\, 0\rangle \\ j\,(\chi)\,S'\,T' \end{matrix} \right\rangle$ of $V$ in $\langle\mu\ 1\ 0\rangle$ when $\chi$ is even.

| | $i\,(\chi)\,S\,T$ | |
|---|---|---|
| $j\,(\chi')\,S'\,T'$ | $1\,(\chi)\,S\,T$ | $2\,(\chi)\,S\,T$ |
| $1\,(\chi-2)\,S+1\ T+1$ | $\left(\frac{S(S+2)(T+2)(\chi-2)(2\mu+6-\chi)\,W(\chi,S,T)}{(S+1)\,W(\chi-2,S+1,T+1)}\right)^{1/2}$ | 0 |
| $2\,(\chi-2)\,S+1\ T+1$ | 0 | $\left(\frac{(S+2)T(T+2)(\chi-2)(2\mu+6-\chi)\,W(\chi,T,S)}{(T+1)\,W(\chi-2,T+1,S+1)}\right)^{1/2}$ |
| $1\,(\chi)\,S+1\ T-1$ | $\left(\frac{S(S+2)(T+1)(2S+\chi+1)(2T+\chi+1)\,W(\chi,S+1,T-1)}{(S+1)\,W(\chi,S,T)}\right)^{1/2}$ | $\{(2S+\chi+1)(2T+\chi+1)[ST(S+T+3)+2T-(S+1)^2]$ $+(2S+1)(T-S-1)[T(2T+1)-(S+1)]\}$ $\times\left(\frac{(S+2)(T+1)}{(S+1)\,TW(\chi,T,S)\,W(\chi,S+1,T-1)}\right)^{1/2}$ |
| $2\,(\chi)\,S+1\ T-1$ | $(S+T+2)[S(S+2)(T-1)(T+1)]^{1/2}$ $\times\left(\frac{(2S+\chi+1)(2S+\chi+3)(2T+\chi-1)(2T+\chi+1)}{W(\chi,S,T)\,W(\chi,T-1,S+1)}\right)^{1/2}$ | $\left(\frac{(S+2)(T-1)(T+1)(2S+\chi+3)(2T+\chi-1)\,W(\chi,T,S)}{TW(\chi,T-1,S+1)}\right)^{1/2}$ |
| $1\,(\chi-1)\,S+1\ T$ | 0 | $(S+T+2)\left(\frac{S(2T+1)(2S+\chi+1)(2T+\chi+1)}{(T+1)\,W(\chi,S,T)}\right)^{1/2}$ |
| $2(\chi-1)\,S+1\ T$ | 0 | $\left(\frac{(2T+1)\,W(\chi,T,S)}{T(T+1)}\right)^{1/2}$ |
| $1\,(\chi-1)\,S\,T+1$ | $\left(\frac{(2S+1)\,W(\chi,S,T)}{S(S+1)}\right)^{1/2}$ | 0 |
| $2\,(\chi-1)\,S\,T+1$ | $(S+T+2)\left(\frac{(2S+1)T(2S+\chi+1)(2T+\chi+1)}{(S+1)\,W(\chi,T,S)}\right)^{1/2}$ | 0 |

**Table 4.** Reduced matrix elements $\left\langle \begin{matrix} \langle \mu\, 1\, 0 \rangle \\ i\,(\chi)\, S\, T \end{matrix} \middle\| V \middle\| \begin{matrix} \langle \mu\, 1\, 0 \rangle \\ j\,(\chi')\, S'\, T' \end{matrix} \right\rangle$ of $V$ in $\langle \mu\, 1\, 0 \rangle$ when $\chi$ is odd.

| | $i\,(\chi)\,S\,T$ | |
|---|---|---|
| $j\,(\chi')\,S'\,T'$ | $1\,(\chi)\,S\,T$ | $2\,(\chi)\,S\,T$ |
| $1\,(\chi-2)\,S+1\;T+1$ | $\left(\frac{S(S+2)(T+1)(\chi-1)(2\mu+5-\chi)}{(S+1)}\right)^{1/2}$ | 0 |
| $2\,(\chi-2)\,S+1\;T+1$ | 0 | $\left(\frac{T(T+2)(S+1)(\chi-1)(2\mu+5-\chi)}{(T+1)}\right)^{1/2}$ |
| $1\,(\chi)\,S+1\;T-1$ | $\left(\frac{S(S+2)T(2S+\chi+2)(2T+\chi)}{(S+1)}\right)^{1/2}$ | 0 |
| $2\,(\chi)\,S+1\;T-1$ | 0 | $\left(\frac{(S+1)(T-1)(T+1)(2S+\chi+2)(2T+\chi)}{T}\right)^{1/2}$ |
| $1\,(\chi)\,S\,T$ | 0 | $-[(2S+1)(2T+1)]^{1/2}$ |
| $2\,(\chi)\,S\,T$ | $-[(2S+1)(2T+1)]^{1/2}$ | 0 |
| $1\,(\chi-1)\,S+1\;T$ | 0 | $(T-S)\left(\frac{(S+2)(2T+1)(\chi-1)(2\mu+5-\chi)}{(T+1)\,W(\chi-1,\,S+1,\,T)}\right)^{1/2}$ |
| $2\,(\chi-1)\,S+1\;T$ | 0 | $-\left(\frac{(S+1)(S+2)(2T+1)(2S+\chi+2)(2T+\chi)(\chi-1)(2\mu+5-\chi)}{T(T+1)\,W(\chi-1,\,T,\,S+1)}\right)^{1/2}$ |
| $1\,(\chi-1)\,S\;T+1$ | $-\left(\frac{(2S+1)(T+1)(T+2)(2S+\chi)(2T+\chi+2)(\chi-1)(2\mu+5-\chi)}{S(S+1)\,W(\chi-1,\,S,\,T+1)}\right)^{1/2}$ | 0 |
| $2\,(\chi-1)\,S\;T+1$ | $(S-T)\left(\frac{(2S+1)(T+2)(\chi-1)(2\mu+5-\chi)}{(S+1)\,W(\chi-1,\,T+1,\,S)}\right)^{1/2}$ | 0 |

So there are only singlet states

$$\left| \begin{matrix} \langle \mu\ 1\ 0 \rangle \\ 1\ (\chi)\ S\ 0 \end{matrix} \right\rangle \qquad \text{or} \qquad \left| \begin{matrix} \langle \mu\ 1\ 0 \rangle \\ 2\ (\chi)\ 0\ T \end{matrix} \right\rangle$$

for $T = 0$ or $S = 0$, respectively.

When $\chi$ is even and non-zero, the ISF are also given in table 1. In this case, the states

$$\left| \begin{matrix} \langle \mu\ 1\ 0 \rangle \\ k\ (\chi = \text{even})\ S\ T \end{matrix} \right\rangle$$

are normalised but are not orthogonal for different $k$:

$$\begin{aligned} \rho_{11} &= \rho_{22} = 1, \\ \rho_{12} &= \rho_{21} = (S+T+2)\left(\frac{ST(2S+\chi+1)(2T+\chi+1)}{W(\chi, S, T)\,W(\chi, T, S)}\right)^{1/2} \end{aligned} \tag{5.4}$$

where

$$W(\chi, S, T) = S(2T+1)(S+T+2) + \chi(T+1)(2\mu+4-\chi).$$

Using the above ISF we get the reduced matrix elements of $V$ from (4.9) and (4.9′). The results are given in tables 3 and 4. Then we obtain the explicit matrix elements of generators in the IR $\langle \mu\ 1\ 0 \rangle$.

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